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A class of stationary Einstein–Maxwell solutions with cylindrical symmetry

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Abstract. The field equations for a stationary cylindrically symmetric electrovac space–time, having a space-like hypersurface-orthogonal Killing field, are rederived using the Kinnersley–Chitre formalism, using the additional assumption that the only surviving components of the electromagnetic potential are $A_t(r)$ and $A_\phi(r)$. New families of solutions are presented for a non-null electromagnetic field. One particular family can be completely described in terms of Painlevé transcendents. The resulting space–times are not static. All possible locally static cylindrically symmetric Einstein–Maxwell solutions of the considered type are listed.

1. Introduction

Current research in stationary axially symmetric Einstein–Maxwell solutions is mainly directed to generation techniques for asymptotically flat space–times. This partially explains the lack of exact solutions when space–time is also endowed with cylindrical symmetry. Although one deals in the latter case with a system of ordinary rather than partial differential equations, the only stationary example in the survey of Kramer *et al* (1980) is the Wilson solution (1968), which was shown by McCrea (1982) not to be an electrovac solution at all. The stationary cylindrical solutions of Arbex and Som (1973) are in fact known to be static, and the only stationary solutions which are not locally static are the McCrea solutions (1982), describing gravity coupled with a null electromagnetic field.

It is the purpose of this paper to present new solutions for a non-null electromagnetic field. In § 2 the field equations are derived in the Kinnersley–Chitre formalism (1977), as this is best suited to obtain directly first integrals for some of the equations. This yields a set of two coupled nonlinear second-order ordinary differential equations, of which new families of solutions are given in § 3. In § 4 we derive general criteria for a stationary cylindrically symmetric electrovac space–time of the considered type to possess a time-like hypersurface-orthogonal Killing field. It follows that the new solutions are not locally static.

2. Field equations

We write the metric of the stationary cylindrically symmetric space–time having a space-like hypersurface-orthogonal Killing field in the Weyl–Lewis–Papapetrou form

$$ds^2 = f(dt - \omega\, d\phi)^2 - f^{-1}[e^{2\gamma}(dr^2 + dz^2) + r^2\, d\phi^2] \tag{2.1}$$

with f, ω and γ functions of r only.

Denoting the coordinates t, ϕ, r, z by x^1, x^2, x^3 and x^4 respectively, we define $f_{11} = f$, $f_{12} = -f\omega = f_{21}$ and $f_{22} = -r^2 f^{-1} + \omega^2 f$.

We also assume that the only non-vanishing components of the electromagnetic potential are $A_1 = P(r)$ and $A_2 = Q(r)$.

According to the work of Kinnersley and Chitre (1977), the Einstein–Maxwell equations

$$R_{ab} = -2(F_a{}^c F_{bc} - \tfrac{1}{4} g_{ab} F_{cd} F^{cd}) \tag{2.2}$$

with

$$F_{ab} = A_{b,a} - A_{a,b} \tag{2.3}$$

fall into two classes, the first one of which gives γ by integration from f, ω and P, and the second reduces to the Ernst equations

$$f\nabla^2\Phi = (\nabla G + 2\Phi^*\nabla\Phi)\cdot\nabla\Phi, \tag{2.4}$$

$$f\nabla^2 G = (\nabla G + 2\Phi^*\nabla\Phi)\cdot\nabla G. \tag{2.5}$$

G and Φ are complex functions of r and z, with $G = G_{11}$, $\Phi = \Phi_{11}$ and $(C, D, \ldots = 1, 2)$

$$G_{CD} = f_{CD} - \Phi_C^*\Phi_D + i\Psi_{CD} + 2iA_C B_D, \tag{2.6}$$

$$\Phi_C = A_C + iB_C. \tag{2.7}$$

B_C and Ψ_{CD} are potentials defined by

$$\nabla B_C = -r^{-1} f_C{}^D \tilde{\nabla} A_D, \tag{2.8}$$

$$\nabla\Psi_{CD} = -r^{-1}(f_C{}^E \tilde{\nabla} f_{ED} - 2f_C{}^E A_D \tilde{\nabla} A_E - 2f_D{}^E A_C \tilde{\nabla} A_E) \tag{2.9}$$

(∇, $\tilde{\nabla}$ being the operators (∂_3, ∂_4) and $(\partial_4, -\partial_3)$ respectively, and indices being raised with the alternating symbol $\varepsilon^{CD} = \varepsilon_{CD} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$).

Imposing now the conditions for cylindrical symmetry $A_C = A_C(r)$ and $f_{CD} = f_{CD}(r)$, one deduces from (2.8) and (2.9)

$$B_C = b_C z + c_C, \tag{2.10}$$

$$\Psi_{CD} = y_{CD} z + v_{CD}, \tag{2.11}$$

with b_C, c_C, y_{CD} and v_{CD} constants. By suitable gauge transformations one can take $c_C = v_{CD} = 0$. Substitution in the inverse relations of (2.8) and (2.9) shows that b_C and y_{CD} are real, with

$$A'_C = r^{-1} f_C{}^D b_D, \tag{2.12}$$

$$f'_{CD} = r^{-1} f_C{}^E (y_{ED} + 2A_E b_D + 2A_D b_E) \tag{2.13}$$

(a prime denoting differentiation with respect to r). From (2.8) and (2.9) one also obtains

$$\tilde{\nabla} B_1 = -r^{-1} f(\nabla A_2 + \omega \nabla A_1) \tag{2.14}$$

and

$$\tilde{\nabla}\omega = -rf^{-2}(\nabla\Psi_{11} + 4A_1 \nabla B_1). \tag{2.15}$$

Hence, (2.10)–(2.13) yield

$$Q' = -qrf^{-1} - \omega P' \tag{2.16}$$

and

$$\omega' = rf^{-2}(c + 4qP) \tag{2.17}$$

where we have put $q = b_1$ and $c = y_{11}$.

With $\Phi = P + iqz$ and $G = f + \Phi^2 - 2P^2 + icz$ the Ernst equations (2.4) and (2.5) now become

$$(rf^{-1}P')' = -qrf^{-2}(c + 4qP) \tag{2.18}$$

and

$$(rf^{-1}f')' = rf^{-2}[2f(P'^2 + q^2) - (c + 4qP)^2]. \tag{2.19}$$

Once f and P have been calculated from this system, Q and ω follow from (2.16) and (2.17), and finally the metric coefficient γ results from the integration of

$$\gamma' = \tfrac{1}{4}rf^{-2}f'^2 - \tfrac{1}{4}r^{-1}f^2\omega'^2 + q^2rf^{-1} - rf^{-1}P'^2 \tag{2.20}$$

(cf Kramer *et al* 1980). Note also that one obtains from (2.17) and (2.18)

$$rf^{-1}P' = p - q\omega \tag{2.21}$$

with p a constant.

3. Solutions

3.1. $q = c = 0$

From (2.16) and (2.17) one sees that the resulting solutions are static (ω = constant) and have magnetic and electrostatic potentials proportional to each other (up to a trivial integration constant):

$$Q = -\omega P. \tag{3.1}$$

In this sense they can be looked at as superpositions of the Witten solutions (1962). For P a constant they reduce to the vacuum solutions (cf Kramer *et al* 1980). When $P' \neq 0$, the solutions can be obtained explicitly by rewriting (2.19) and (2.21) as

$$\mathrm{d}^2f/\mathrm{d}P^2 = 2, \tag{3.2}$$

$$\mathrm{d}\ln r/\mathrm{d}P = p^{-1}f^{-1}. \tag{3.3}$$

The general solution has three arbitrary constants, and contains as an important subclass the solutions (v, w constants)

$$f = v^2 \sec^2(pv \ln wr), \qquad P = v \tan(pv \ln wr), \tag{3.4}$$

and

$$f = v^2 \operatorname{cosech}^2(pv \ln wr), \qquad P = -v \coth(pv \ln wr) \tag{3.5}$$

for which the Ernst potential G is a constant (such that (2.5) is trivially satisfied).

3.2. $q = 0, c \neq 0$

Non-trivial electrovac solutions with Q and P proportional to each other, or with a null electromagnetic field, do not exist. Rewriting (2.19) and (2.21) as differential equations in P yields

$$d^2f/dP^2 = 2 - c^2p^{-2}r^2f^{-3}, \tag{3.6}$$

$$d\ln r/dP = p^{-1}f^{-1}. \tag{3.7}$$

One obtains then from (2.16) and (2.17)

$$\omega = -dQ/dP \tag{3.8}$$

and

$$Q = pc^{-1}(f - P^2). \tag{3.9}$$

A particular class of solutions results by taking f proportional to P. One has then

$$f = (2p^2c^{-2})^{-1/3}r^{2/3}, \qquad P = \tfrac{3}{2}pf, \qquad \omega = c^{-1}(3p^2f - \tfrac{2}{3}). \tag{3.10}$$

The metric coefficient γ follows from (2.20):

$$\gamma = \tfrac{1}{9}\ln r - \tfrac{9}{4}p^2(2p^2c^{-2})^{-1/3}r^{2/3}. \tag{3.11}$$

Starting with this particular solution $f_1 = (2p^2c^{-2})^{-1/3}r^{2/3}$, we can construct all other $q = 0$ solutions: (2.21) yields

$$P' = pr^{-1}f \tag{3.12}$$

with f given by (2.19), or

$$(rf'/f)' = 2p^2r^{-1}f - c^2rf^{-2}. \tag{3.13}$$

Define

$$f = f_1 \cdot u$$

and

$$x = (\tfrac{3}{2})^{3/2}r^{1/2}(2p^2|c|)^{1/2}. \tag{3.14}$$

Then (3.13) yields

$$(d/dx)(xu^{-1}\,du/dx) = (\tfrac{4}{3})^2x^{1/3}(u - u^{-2}). \tag{3.15}$$

The substitution $u = x^{-1/3}y^{-1}$ then finally gives y as the third Painlevé-transcendent:

$$d^2y/dx^2 = y^{-1}(dy/dx)^2 - x^{-1}\,dy/dx + \tfrac{16}{9}(y^3 - x^{-1}). \tag{3.16}$$

It follows that (3.10) is the unique solution for $q = 0$ and $c \neq 0$, which is solvable in terms of classical transcendental and elementary functions.

We shall show in § 4 that all the $q = 0$ solutions are stationary and non-static.

3.3. $q \neq 0$

(i) Non-trivial solutions for which $(Q - kP)' = 0$ exist. Elimination of P' from (2.16) and (2.21) yields

$$\omega = \tfrac{1}{2}(p - kq)q^{-1} \pm [r^2f^{-2} + \tfrac{1}{4}q^{-2}(p + kq)^2]^{1/2}. \tag{3.17}$$

In general, an arbitrary relation $\omega=\omega(r,f)$ will be incompatible with (2.18)–(2.19). However, for (3.17) the integrability conditions resulting from (2.17), (2.21) are precisely equivalent with (2.19). Hence, each solution of (2.19), i.e.

$$(rf^{-1}f')'=2rf^{-1}q^2+2r^{-1}f(p-q\omega)^2-r^{-1}f^2\omega'^2 \tag{3.18}$$

with ω given by (3.17), yields a solution of the system (2.17)–(2.19).

When $p+kq=0$, one has $\omega=-k\pm rf^{-1}$ and hence

$$P=\mp qr, \qquad Q=\pm pr. \tag{3.19}$$

Integration of (3.18) yields further

$$f=4q^2r^2+vr\ln wr \qquad (v,\ w \text{ constants}) \tag{3.20}$$

in which one recognises the null solutions given by McCrea (1982). The electromagnetic field in the orthonormal basis of one-forms is then

$$F=2q\ \mathrm{e}^{-\gamma}(\omega^2\wedge\omega^3\pm\omega^1\wedge\omega^3) \tag{3.21}$$

with

$$\begin{aligned} \omega^1&=f^{1/2}(\mathrm{d}t-\omega\ \mathrm{d}\phi), & \omega^3&=f^{-1/2}\ \mathrm{e}^{\gamma}\ \mathrm{d}r,\\ \omega^2&=f^{-1/2}r\ \mathrm{d}\phi, & \omega^4&=f^{-1/2}\ \mathrm{e}^{\gamma}\ \mathrm{d}z. \end{aligned} \tag{3.22}$$

When $p+kq\neq 0$, a particular one-parameter family of solutions for (3.18) has been given by Arbex and Som (1973). The solutions are of electrostatic or magnetostatic type respectively,

$$f=\cosh^2 u[(r^v+mr^{-v})^{-2}-(\tanh^2 u)r^2(r^v+mr^{-v})^2]$$

and

$$f=\cosh^2 u[(r^{1+w}-nr^{1-w})^2-(\tanh^2 u)r^2(r^{1+w}-nr^{1-w})^{-2}] \tag{3.23}$$

(v, w are constants of integration; m and n determined by p, q and k). They describe a static field viewed by a rotating observer. In § 4 we will show that all other solutions, resulting from (3.17) with $p+kq\neq 0$, are locally static too. The hypersurface orthogonal time-like Killing fields are $-k\,\partial/\partial t+\partial/\partial\phi$ or $(2p+kq)\partial/\partial t+q\,\partial/\partial\phi$ according to the choice of the upper or lower sign in (3.17) when $(p+qk)q>0$ (and the opposite when $(p+qk)q<0$).

(ii) Substitution in (2.18)–(2.19) of the ansatz $P=vr^{m+1}+w$ (v; w and m constants) leads unambiguously to $m=0$ or $m=\frac{1}{3}$. For $m=0$ one reobtains the McCrea null solutions, while $m=\frac{1}{3}$ yields

$$P=vr^{4/3}-\tfrac{1}{4}cq^{-1} \tag{3.24}$$

and

$$f=\tfrac{9}{2}q^2r^2-\tfrac{1}{2}(\tfrac{3}{2}q)^4v^{-2}r^{4/3}. \tag{3.25}$$

Hence, (2.20) and (2.21) yield

$$\omega=q^{-1}(p-\tfrac{4}{3}vr^{4/3}f^{-1}) \tag{3.26}$$

and

$$\gamma=\ln[r^{4/9}(\tfrac{16}{9}v^2q^{-2}r^{2/3}-1)^{1/2}]-\tfrac{8}{9}v^2q^{-2}r^{2/3}. \tag{3.27}$$

Below, we will show that these solutions are non-static too.

4. Criteria for the considered stationary cylindrically symmetric electrovac space–time to be static

It is well known that, with a suitable choice of coordinates, the metric of a static axisymmetric electrovac space–time can be written—at least locally—in the form (2.1) with $\omega = 0$. Whether such a coordinate system exists which covers the whole space–time, such that the solution is *globally* static, will not be our point of study. For an introduction to this subject, one is referred to Stachel (1982).

Obviously, when ω is constant, solutions are also locally static: it suffices to look at the coordinate transformation $t' = t - \omega\phi$. In general, for non-constant ω, it is hard to find out whether such a transformation exists. For space–times endowed with cylindrical symmetry too, it was claimed by Som *et al* (1976) that any stationary vacuum field solution is necessarily static. This claim however turned out to be false (Bonnor 1980).

Neither for an electrovac space–time is this the case, as has been proved by McCrea for the null solutions (3.19)–(3.20).

We will develop now general criteria for stationary cylindrically symmetric electrovac space–times of the considered type to possess a time-like hypersurface-orthogonal Killing field X. Obviously one does not know *a priori* whether the three Killing fields $\partial/\partial x^1$, $\partial/\partial x^2$ and $\partial/\partial x^4$ span the Lie algebra of the complete isometry group (this is e.g. not true for the flat solution $f = 1$, $\gamma = \omega = 0$). Therefore we first solve the Killing equations

$$X_{(a;b)} = 0 \tag{4.1}$$

to find out whether there are other solutions besides the obvious ones $X^i = \text{constant}$, $X^3 = 0$ ($i = 1, 2, 4$). Defining $g = f^{-1/2}\,e^{\gamma}$, one obtains from (4.1)

$$X^3{}_{,3} = X^4{}_{,4} = g^{-1}g'X^3, \tag{4.2}$$

$$X^3{}_{,4} + X^4{}_{,3} = 0, \tag{4.3}$$

$$X^1{}_{,1} = \tfrac{1}{2}r^{-2}f_{22}f'X^3, \tag{4.4}$$

$$X^2{}_{,2} = \tfrac{1}{2}r^{-2}ff'_{22}X^3, \tag{4.5}$$

$$fX^1{}_{,2} + f_{22}X^2{}_{,1} - r^{-2}(rf_{12} + ff_{22}f'_{12})X^3 = 0, \tag{4.6}$$

$$X^3{}_{,2} + \omega X^3{}_{,1} - r^2\,e^{-2\gamma}X^2{}_{,3} = 0, \tag{4.7}$$

$$X^4{}_{,2} + \omega X^4{}_{,1} - r^2\,e^{-2\gamma}X^2{}_{,4} = 0, \tag{4.8}$$

$$f_{22}X^3{}_{,1} - f_{12}X^3{}_{,2} - r^2g^{-2}X^1{}_{,3} = 0, \tag{4.9}$$

$$f_{22}X^4{}_{,1} - f_{12}X^4{}_{,2} - r^2g^{-2}X^1{}_{,4} = 0. \tag{4.10}$$

We limit ourselves now to the cases for which:

(i) $\omega \neq$ constant (then space–time would be static);

(ii) $f \neq$ constant (all solutions are then necessarily flat);

(iii) $\omega \neq Krf^{-1}$, K constant (one obtains then the null solutions ($K = \pm 1$) which are non-static (McCrea 1982));

(iv) $\omega^2 \neq K + r^2f^{-2}$, K constant, or, equivalently $(f^{-1}f_{12})' \neq 0$ (these solutions will be shown below to be static with hypersurface-orthogonal Killing field $n_0\partial/\partial x^1 + n_1\partial/\partial x^2$).

From (4.2) one has now

$$X^3 = \psi(x^1, x^2, x^4)\cdot g. \tag{4.11}$$

(a) When $\psi = 0$, (4.2)–(4.5) and (4.7)–(4.9) yield $X^4 = X^4(x^1, x^2)$, $X^1 = X^1(x^2, x^4)$ and $X^2 = X^2(x^1, x^4)$. Taking x^1, x^2 partial derivatives of the remaining equations and making use of the conditions (i)–(iv) shows that X^1, X^2 and X^4 are linear in their arguments with $X^4{}_{,2} = 0$. Substituting back in the equations yields X^1, X^2 and $X^4 = \text{constant}$.

(b) When $\psi \neq 0$, (4.2)–(4.3) and (4.11) yield ($r = x^3$, $z = x^4$)

$$\begin{aligned} X^3 &= (a_1\,\mathrm{e}^{\mathrm{i}Kr} + a_2\,\mathrm{e}^{-\mathrm{i}Kr})(b_1\,\mathrm{e}^{Kz} + b_2\,\mathrm{e}^{-Kz}),\\ X^4 &= (a_1\,\mathrm{e}^{\mathrm{i}Kr} - a_2\,\mathrm{e}^{-\mathrm{i}Kr})(b_1\,\mathrm{e}^{Kz} - b_2\,\mathrm{e}^{-Kz}), \end{aligned} \tag{4.12}$$

or

$$\begin{aligned} X^3 &= (a_1 r + a_2)(b_1 z + b_2),\\ X^4 &= \tfrac{1}{2}b_1(a_1 r^2 + 2a_2 r) - \tfrac{1}{2}a_1(b_1 z^2 + 2b_2 z) + b_3, \end{aligned} \tag{4.13}$$

with a_1, a_2 and K constants (K real or imaginary), and $b_i = b_i(x^1, x^2)$. Substituting back in the remaining equations shows that $a_i \neq 0$ and $b_i \neq 0$ are incompatible with conditions (i)–(iv).

For our search for hypersurface-orthogonal time-like Killing fields, this means that we only have to consider fields of the form

$$X = n_0\,\partial/\partial t + n_1\,\partial/\partial\phi + n_2\,\partial/\partial z. \tag{4.14}$$

A Killing field X is hypersurface orthogonal if the associated one-form

$$V = -|X^c X_c|^{-1} g_{ab} X^a\,\mathrm{d}x^b \tag{4.15}$$

is closed. Hence, we must look for cylindrically symmetric stationary electrovac space–times, for which

$$\mathrm{d}[\![\rho^{-2}\{f(n_0 - \omega n_1)\,\mathrm{d}t - [n_1 r^2 f^{-1} + (n_0 - \omega n_1)\omega f]\,\mathrm{d}\phi - n_2 f^{-1}\,\mathrm{e}^{2\gamma}\,\mathrm{d}z\}]\!] = 0 \tag{4.16}$$

and

$$\rho^2 > 0$$

with

$$\rho^2 = X^a X_a = f(n_0 - \omega n_1)^2 - f^{-1}(n_1^2 r^2 + n_2^2\,\mathrm{e}^{2\gamma}). \tag{4.17}$$

We can restrict ourselves to the case $n_0 - \omega n_1 \neq 0$ ($n_0 - \omega n_1 = 0$ would imply $\omega =$ constant or $n_0 = n_1 = 0$ and hence $\rho^2 \leqslant 0$). Now (4.16) yields

$$\begin{aligned} &(\rho^2)' f(n_0 - \omega n_1) - \rho^2[f(n_0 - \omega n_1)]' = 0,\\ &(\rho^2)'[r^2 n_1 f^{-1} + \omega(n_0 - \omega n_1) f] - \rho^2[r^2 n_1 f^{-1} + \omega(n_0 - \omega n_1) f]' = 0,\\ &(\rho^2)'[n_2\,\mathrm{e}^{2\gamma} f^{-1}] - \rho^2[n_2\,\mathrm{e}^{2\gamma} f^{-1}]' = 0. \end{aligned} \tag{4.18}$$

Hence, constants λ, μ and ν ($\lambda \neq 0$) exist such that

$$\rho^2 = \lambda f(n_0 - \omega n_1), \tag{4.19}$$

$$\mu\rho^2 = n_1 r^2 f^{-1} + \omega(n_0 - \omega n_1) f, \tag{4.20}$$

$$\nu\rho^2 = n_2\,\mathrm{e}^{2\gamma} f^{-1}. \tag{4.21}$$

(a) $\nu = n_2 = 0$
Equations (4.17) and (4.19)–(4.20) can be seen to be incompatible, except for $n_1 \neq 0$ and $\mu = (n_0 - \lambda)(\lambda n_1)^{-1}$. One has then

$$\omega = n_0/n_1 - \kappa \pm (r^2 f^{-2} + \kappa^2)^{1/2} \tag{4.22}$$

with $\kappa = \lambda/2n_1$. Eliminating P and P' with (4.22) from (2.17) and (2.21), a differential equation for f results which is identical with (2.19) for

$$p = q n_0 n_1^{-1} \qquad \text{or} \qquad p = q(n_0 - \lambda) n_1^{-1}. \tag{4.23}$$

Hence $q \neq 0$ (if not, $p = 0$, and one would have vacuum solutions).

One can check that the Killing field is time-like ($\rho^2 > 0$) when the following choice of signs is made in (4.22):

$$\omega = n_0/n_1 - \kappa - \kappa(1 + \kappa^{-2} r^2 f^{-2})^{1/2}. \tag{4.24}$$

For any value of $\kappa\,(\neq 0)$, two independent Killing fields exist (cf (4.23)).

It also follows that all these static solutions belong to the class studied in § 3.3 (i): $(Q - kP)' = 0$ with k given by $k = \pm 2\kappa - p/q$, according to the choice (4.23) of the Killing field.

(b) $\nu \neq 0$
(4.17) and (4.19)–(4.21) yield $n_1 \neq 0$ and $n_0 \neq \lambda\mu$. One obtains then explicit expressions for $\omega(r)$ and $f(r)$:

$$f = \alpha + \beta r^2, \qquad \omega = (\zeta + \xi r^2) f^{-1},$$

with α, β, ζ and ξ constants determined by λ, μ, n_0, n_1 and n_2. Substitution of these in (2.18)–(2.19) shows that the only solution of this type is the flat solution $\beta = \zeta = \xi = 0$.

5. Conclusion

Some new families of exact solutions for a cylindrically symmetric stationary space–time with a non-null electromagnetic field have been presented: (3.4), (3.10)–(3.11), (3.16) and (3.24)–(3.27). The former is static and has constant Ernst potential, the others are not locally static. Furthermore, it has been shown that the considered cylindrically symmetric stationary electrovac space–times, with non-vanishing electromagnetic field, are locally static if and only if ω is a constant *or* the solution is of the type (4.24) with $(Q - kP)' = 0$.

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